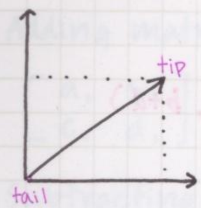
MEMO

MULTIVARIABLE
CALCULUS

VECTORS

9.12.23

- has **magnitude** $|\vec{v}|$ and **direction** $\frac{\vec{v}}{|\vec{v}|}$



$\vec{v} = \langle 4, 3 \rangle = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$= 4\hat{i} + 3\hat{j}$

UNIT VECTORS

$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- unit vector in the direction of $\vec{v} = \langle a, b \rangle$:

$$\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right) \leftarrow \frac{\vec{v}}{\underbrace{|\vec{v}|}_{\text{magnitude}}}$$

MAGNITUDE AND DIRECTION

- magnitude of $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$:

$$\sqrt{a^2 + b^2 + c^2} = |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$$

- direction cosines of $\vec{v}$:

$$\left(\frac{a}{|\vec{v}|}, \frac{b}{|\vec{v}|}, \frac{c}{|\vec{v}|} \right) = (\cos\alpha, \cos\beta, \cos\gamma)$$

angle $\vec{v}$ makes w/ axes

$\alpha = \cos^{-1}\left(\frac{a}{|\vec{v}|}\right)$ x-axis

$\beta = \cos^{-1}\left(\frac{b}{|\vec{v}|}\right)$ y-axis

$\gamma = \cos^{-1}\left(\frac{c}{|\vec{v}|}\right)$ z-axis

COLLINEAR

- $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ is collinear to:

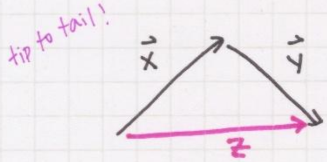
$$x(a\hat{i} + b\hat{j} + c\hat{k}) = ax\hat{i} + bx\hat{j} + cx\hat{k}$$

SCALAR MULTIPLICATION

VECTORS

9.12.23

ADDING VECTORS



$\vec{x} + \vec{y} = \vec{z}$

$(a,b) + (c,d) = (a+c\ ,\ b+d)$

PARALLEL

- vectors are parallel if their <u>unit vectors</u> are <u>parallel</u>

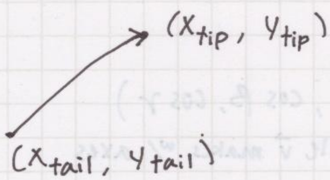
$$\hat{u} = \pm \hat{v}$$

- or if there is a scalar $s \neq 0$ such that

$$\vec{a} = s \cdot \vec{b}$$

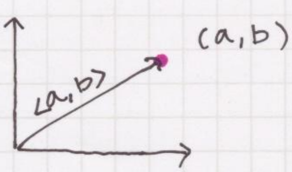
- **dimension**: # of entries
- **component**: each individual entry

VECTOR BETWEEN POINTS



$\vec{v} = (x_{tip} - x_{tail}\ ,\ y_{tip} - y_{tail})$

STANDARD POSITION



- a vector w/ its tail @ the origin

MATRICES

9.12.23

ADDING AND SUBTRACTING

- Adding matrices:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

- Subtracting matrices:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + (-1)\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{bmatrix}$$

- CANNOT add/subtract matrices w/ different dimensions!

MULTIPLYING

- Multiplying matrices by scalars:

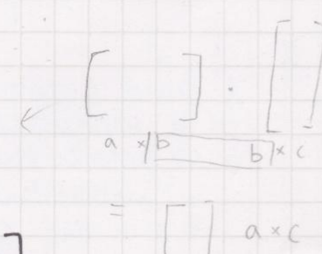
$$x \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax & bx \\ cx & dx \end{bmatrix}$$

- Multiplying matrices:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} (a_1)(a_2) + (b_1)(c_2) & (a_1)(b_2) + (b_1)(d_2) \\ (c_1)(a_2) + (d_1)(c_2) & (c_1)(b_2) + (d_1)(d_2) \end{bmatrix}$$

- Multiplying matrices w/ different dimensions:
  - ↳ only works if # of X's columns = # of Y's rows!
  - ↳ Order Matters! $X \cdot Y \neq Y \cdot X$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \\ e_2 & f_2 \end{bmatrix}$$



$$= \begin{bmatrix} \text{dot product} & \text{dot product} \\ \text{dot product} & \text{dot product} \end{bmatrix}$$

DOT PRODUCT

$$[a_1 \ b_1 \ c_1] \cdot [a_2 \ b_2 \ c_2] = (a_1)(a_2) + (b_1)(b_2) + (c_1)(c_2)$$

MATRICES

9.12.23

DETERMINANTS

- determinant of 2×2 matrix: det [matrix]

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (a)(d) - (b)(c)$$

= |matrix|

- determinant of 3×3 matrix:

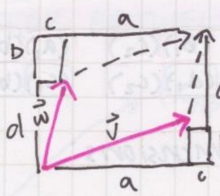
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = (a)(e)(i) + (b)(f)(g) + (c)(d)(h) - (c)(e)(g) - (a)(f)(h) - (b)(d)(i)$$

OR

$$\begin{vmatrix} x_c & y_c & z_c \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} x_c - \begin{vmatrix} x_a & z_a \\ x_b & z_b \end{vmatrix} y_c + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} z_c$$

GEOMETRY OF THE DETERMINANT

for vectors ⟨a, b⟩ and ⟨c, d⟩:



the area of the parallelogram between them is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2D AND 3D

Real number: a number w/ a (possibly infinite) decimal rep.

$f: \mathbb{R} \to \mathbb{R}$: function that maps a real number to a real number

2D

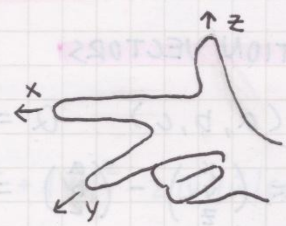
- $\mathbb{R}^2$: set of all ordered pairs of real numbers
- (x, y)
- draws a **curve**

3D

- $\mathbb{R}^3$: set of all ordered triples of real numbers
- (x, y, z)
- draws a **surface**

PLANES

(x, y)-plane: $z = 0$
(x, z)-plane: $y = 0$
(y, z)-plane: $x = 0$



RIGHT
HAND
RULE

DISTANCE FORMULA

$$\mathbb{R}^2: \sqrt{(x-a)^2 + (y-b)^2}$$

$$\mathbb{R}^3: \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

$$\mathbb{R}^n: \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + \ldots + (x_n-a_n)^2}$$

CIRCLES & SPHERES

$$\mathbb{R}^2: \; r^2 = (x-a)^2 + (y-b)^2$$

$$\mathbb{R}^3: \; r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

$$\mathbb{R}^n: \; r^2 = \sum_{i=1}^{n} (x_i - c_i)^2$$

- look similar to the distance formula
  - ↳ set of points r away from the center
- center @ (a, b), (a, b, c), $(c_1, c_2 \cdots c_n)$

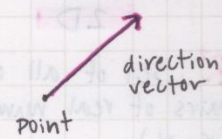
LINES AND PLANES

LINES

$(x, y, z) = (\text{point}) + t(\text{direction vector})$

ex.

$$(x,y,z) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}$$

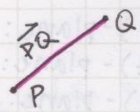


$$\left.\begin{aligned} x &= 1-5t \\ y &= 2-2t \\ z &= 3+3t \end{aligned}\right] \text{parametric}$$

DIRECTION VECTORS

$P = (a, b, c) \qquad Q = (x, y, z)$

$$\overrightarrow{PQ} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix}$$

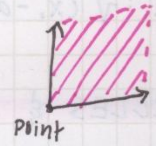


PLANES

$(x, y, z) = (\text{point}) + t(\text{vector}) + s(\text{vector})$

ex.

$$(x,y,z) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$



$$\left.\begin{aligned} x &= 2+t-s \\ y &= -2+2t \\ z &= 1+3t+4s \end{aligned}\right] \text{parametric}$$

- A curve (vector-valued function, maps a real number to a vector) lies on the surface (defined implicitly) if:
  1.) break the curve into components
  2.) substitute x-components for xs, etc.
  3.) if the equation holds, it's true!

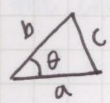
DOT PRODUCT

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boxed{\sum_{i=1}^{n} a_i b_i}$$

vector · vector = number!

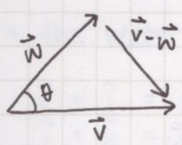
$$= a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

LAW OF COSINES



$$\underline{c^2 = a^2 + b^2 - 2ab\cos(\theta)}$$

With vectors:



$$\underbrace{|\vec{v} - \vec{w}|^2}_{} = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos(\theta)$$

$$(\vec{v}-\vec{w}) \cdot (\vec{v}-\vec{w})$$

$$\vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\vec{v}\cdot\vec{w}$$

$$|\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v}\cdot\vec{w}$$

$$\boxed{\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\theta)}$$

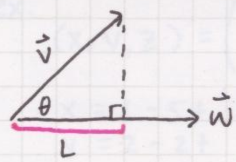
$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right)}$$

- two vectors are **orthogonal** if their dot product is 0
  ↳ $\theta = \pi/2 = 90°$

PROJECTION

SCALAR COMPONENTS

$\text{scal}_{\vec{w}}(\vec{v})$: length of $\vec{v}$ in the direction of $\vec{w}$

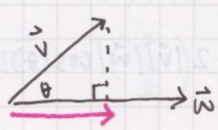


$$L = |\vec{v}| \cdot \cos(\theta)$$

$$= |\vec{v}| \cdot \vec{v} \bullet \vec{w} \cdot \frac{1}{|\vec{v}| \cdot |\vec{w}|} = \boxed{\frac{\vec{v} \bullet \vec{w}}{|\vec{w}|}}$$

PROJECTIONS

$\text{proj}_{\vec{w}}(\vec{v})$: projection of $\vec{v}$ onto $\vec{w}$



magnitude · direction

$$\underbrace{|\vec{v}| \frac{\vec{v} \bullet \vec{w}}{|\vec{v}||\vec{w}|}}_{\text{scal}_{\vec{w}}(\vec{v})} \cdot \underbrace{\frac{\vec{w}}{|\vec{w}|}}_{\text{unit vector of } \vec{w}}$$

$$\boxed{\vec{w} \cdot \frac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}}$$

ORTHOGONAL DECOMPOSITION

$\vec{v}$ in terms of $\vec{w}$:

$$\vec{v} = \underbrace{\text{proj}_{\vec{w}}(\vec{v})}_{\parallel \vec{w}} + \underbrace{(\vec{v} - \text{proj}_{\vec{w}}(\vec{v}))}_{\perp \vec{w}}$$

CROSS PRODUCT

- only exists in $\mathbb{R}^3$ (and $\mathbb{R}^7$???)
- calculated using the determinant

$\langle x_a, y_a, z_a \rangle \times \langle x_b, y_b, z_b \rangle$ ← vector × vector = vector!

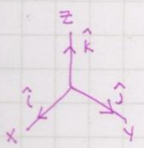
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \hat{i} - \begin{vmatrix} x_a & z_a \\ x_b & z_b \end{vmatrix} \hat{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} \hat{k}$$

- the cross product is **anticommutative**

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

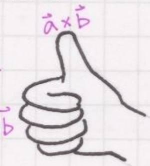
ORTHOGONAL

- the vector $\vec{a} \times \vec{b}$ is <u>orthogonal</u> to both $\vec{a}$ and $\vec{b}$



$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

KRESSER RULE
hand towards $\vec{a}$
curl fingers towards $\vec{b}$



GEOMETRY

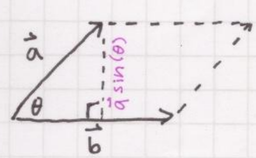
- **Lagrange's Identity**:

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

- using Lagrange's and the geometric interpretation of the dot product:

$$\boxed{|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)}$$

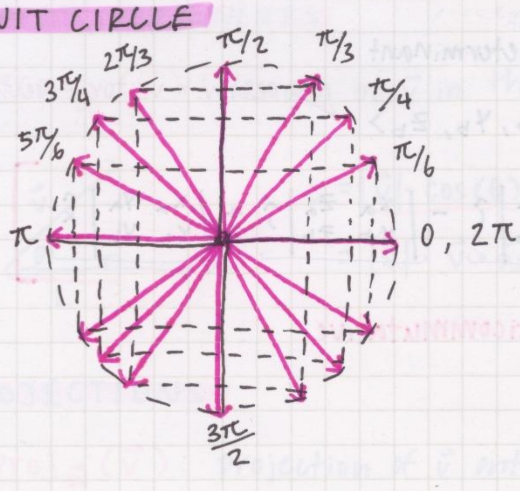
- $|\vec{a} \times \vec{b}|$ also computes the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$:

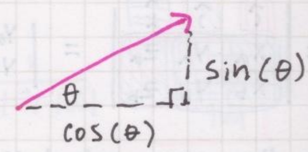


☆ $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\theta = 0$ or $\theta = \pi$

ANGLES

UNIT CIRCLE

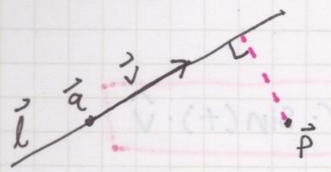


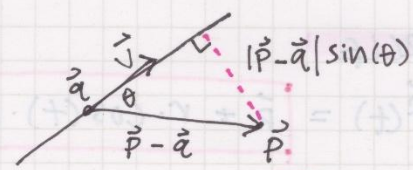


DISTANCES

BETWEEN POINT & LINE

$\vec{l}(t) = \vec{q} + t\vec{v}$





$$\text{distance} = |\vec{p}-\vec{q}|\sin(\theta)$$

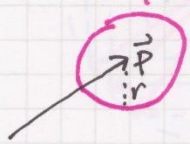
$$= \boxed{\frac{|(\vec{p}-\vec{q}) \times \vec{v}|}{|\vec{v}|}}$$

CIRCLES & ELLIPSES

- given two orthogonal unit vectors $\hat{u}$ and $\hat{v}$ and any other vector $\vec{p}$:

CIRCLE

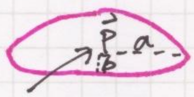
$$\vec{f}(t) = \boxed{\vec{p} + r \cdot \cos(t) \cdot \hat{u} + r \cdot \sin(t) \cdot \hat{v}}$$



- radius r
- centered @ tip of $\vec{p}$
- in the plane of $\hat{u}$ and $\hat{v}$

ELLIPSE

$$\vec{g}(t) = \boxed{\vec{p} + a \cdot \cos(t) \cdot \hat{u} + b \cdot \sin(t) \cdot \hat{v}}$$



- a and b are semi-major and semi-minor axes